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GENERALIZED ORTHOGONALITY RELATION IN THE PROBLEM OF EQUILIBRIUM OF AN ELASTIC CYLINDER

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We derive the generalized orthogonality relation as a generalization of the known relation of Schiff [1, 2] to the case of the nonaxisymmetric deformation of an elastic cylinder; this relation takes place for four variants of the homogeneous boundary conditions at the lateral surfaces of the cylinder. We indicate a method of obtaining the exact solution of the problem by using the introduced relations in the case of two versions, of mixed boundary conditions at the extremities. We compute the values of the generalized norming factors.

1. We write the solution of the differential equations of the eqlasticity theory in displacements in the Papkovich-Neuber form; in the case of cylindrical coordinates $r\varphi z$ we have

$$u_r = u = 4(1-\nu)B_r - \frac{\partial F}{\partial r}, \qquad u_z = w = 4(1-\nu)B_z - \frac{\partial F}{\partial z}$$

$$u_\varphi = v = 4(1-\nu)B_\varphi - \frac{1}{r}\frac{\partial F}{\partial \varphi}, \qquad F = rB_r + zB_z + B_0 \tag{1.1}$$

The functions B_r, B_φ, B_z, B_0 satisfy the differential equations

$$\Delta B_r - \frac{2}{r^2}\frac{\partial B_\varphi}{\partial \varphi} - \frac{B_r}{r^2} = 0, \qquad \Delta B_z = 0$$

$$\Delta B_\varphi + \frac{2}{r^2}\frac{\partial B_r}{\partial \varphi} - \frac{B_\varphi}{r^2} = 0, \qquad \Delta B_0 = 0 \tag{1.2}$$

$$\Delta = \frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial^2}{\partial \varphi^2} + \frac{\partial^2}{\partial z^2}$$

We assume that the prescribed stresses (or displacements) at the lateral surfaces of the cylinder $r = a$, $r = b$ $(a > b)$ are represented by double trigonometric series in the coordinates φ, z (for the continuous cylinder $(b = 0)$ we set a boundedness requirement on the axis). Correspondingly, also the solution is sought in the form of a double trigonometric series. We consider the typical terms of such series

$$u = u^*(r)\cos\lambda z\cos n\varphi, \qquad v = v^*(r)\cos\lambda z\sin n\varphi$$

$$w = w^*(r)\sin\lambda z\cos n\varphi \tag{1.3}$$

for whose determination it is sufficient to take

$$B_r = \frac{\psi(r)}{2(1-\nu)}\cos\lambda z\cos n\varphi, \qquad B_\varphi = \frac{\chi(r)}{2(1-\nu)}\cos\lambda z\sin n\varphi$$

$$B_z = 0, \qquad B_0 = \left[\omega(r) - \frac{r\psi(r)}{2(1-\nu)}\right]\cos\lambda z\cos n\varphi \tag{1.4}$$

From Eqs. (1.2) we obtain the differential relations for the determination of the functions

ψ, χ and ω (the primes denote derivatives with respect to r)

$$r(r\psi')' = (n^2 + 1 + \lambda^2 r^2)\psi + 2n\chi, \qquad r(r\chi')' = (n^2 + 1 + \lambda^2 r^2)\chi + 2n\psi$$

$$r(r\omega')' = (n^2 + \lambda^2 r^2)\omega + \frac{r}{1-\nu}[(r\psi)' + n\chi] \tag{1.5}$$

The functional coefficients u^*, v^*, w^* in the series for the displacements (1.3), by virtue of the expressions (1.1) and (1.4), can be represented in terms of the functions ψ, χ, ω as:

$$u^* = 2\psi - \omega', \qquad v^* = 2\chi + \frac{n}{r}\omega, \qquad \omega^* = \lambda\omega \tag{1.6}$$

In a similar manner we introduce the functional coefficients in the series for the stresses ; their expressions are

$$\sigma_r^* = 2G\left[\frac{\omega'}{r} - \left(\frac{n^2}{r^2} + \lambda^2\right)\omega + r\left(\frac{\psi}{r}\right)' - \frac{n}{r}\chi\right]$$

$$\sigma_\varphi^* = 2G\left[\omega'' - \lambda^2\omega - r\left(\frac{\psi}{r}\right)' + \frac{n}{r}\chi\right]$$

$$\sigma_z^* = 2G\left[\frac{(r\omega')'}{r} - \frac{n^2}{r^2}\omega - \frac{(r\psi)'}{r} - \frac{n}{r}\chi\right]$$

$$\tau_{r\varphi}^* = 2G\left[n\left(\frac{\omega}{r}\right)' - \frac{n}{r}\psi + r\left(\frac{\chi}{r}\right)'\right]$$

$$\tau_{\varphi z}^* = -2G\left(n\frac{\omega}{r} + \chi\right), \qquad \tau_{zr}^* = 2G\lambda(\omega' - \psi) \tag{1.7}$$

In the derivation of the formulas (1.7), for the elimination of Poisson ratio, we have made use of the relation, which follows from the third of the equations (1.5)

$$\frac{\nu}{(1-\nu)r}[(r\psi)' + n\chi] = \frac{(r\omega')'}{r} - \left(\frac{n^2}{r^2} + \lambda^2\right)\omega - \frac{(r\psi)' + n\chi}{r}$$

Solving the system (1.5), we obtain the explicit expressions of the functions ψ, χ and ω

$$\psi = C_1\lambda I_n' - C_2\frac{n}{r}I_n + D_1\lambda K_n' - D_2\frac{n}{r}K_n$$

$$\chi = C_2\lambda I_n' - C_1\frac{n}{r}I_n + D_2\lambda K_n' - D_1\frac{n}{r}K_n \tag{1.8}$$

$$\omega = \frac{r\psi}{2(1-\nu)} + C_3 I_n + D_3 K_n$$

Here $I_n = I_n(\lambda r)$ are the modified Bessel functions, $K_n = K_n(\lambda r)$ are Macdonald functions; C_i, D_i $(i = 1, 2, 3)$ are arbitrary constants which are determined from the conditions on the lateral surfaces of the cylinder for each term of the double trigonometric series. If the length of the cylinder is $2l$, then the values of λ will be $\lambda_m = m\pi/l$ $(m = 0, 1, 2, \ldots)$. In the case of a continuous cylinder we have $D_i = 0$. In a similar manner we can construct the solution for other combinations of the trigonometric functions in the typical terms of the series (1.4).

2. In order to satisfy the boundary conditions at the ends $z = \pm l$ of the cylinder, it is necessary, in general, to solve infinite systems of linear algebraic equations in the arbitrary constants. One method for forming such systems consists in the use of the homogeneous solutions. In this connection, an essential role can be played by the generalized orthogonality relations, similar to those given by Schiff for the case of the axisymmetric deformation of a cylinder [1] and by Papkovich for the plane problem of the theory of elasticity [3]. We will prove that the generalized orthogonality relation exists also in the case of the general problem on the equilibrium of an elastic cylinder.

We consider the following homogeneous boundary conditions at the lateral surfaces of the cylinder: (1) there are no stresses $(\sigma_r = \tau_{r\varphi} = \tau_{rz} = 0)$, (2) there are no displacements $(u = v = w = 0)$, (3) there are no normal displacements and shear stresses $(u = \tau_{r\varphi} = \tau_{rz} = 0)$, (4) there are no normal stresses and tangential displacements $(\sigma_r = v = w = 0)$. From the formulas (1.6) and (1.7) for $r = b$ and $r = a$, respectively, for the cases (1)–(4), we obtain

$$\omega' = \psi, \quad \psi' = n\chi' + \lambda^2\omega, \quad r^2\chi' = r\chi + n\omega \tag{2.1}$$

$$\omega = 0, \quad \chi = 0, \quad \omega' = 2\psi \tag{2.2}$$

$$\omega' = 0, \quad \psi = 0, \quad r^2\chi' = r\chi + n\omega \tag{2.3}$$

$$\omega = 0, \quad \chi = 0, \quad \psi = r\psi' + \omega' \tag{2.4}$$

Inserting the expressions (1.8) for the functions ψ, χ and ω in each of the conditions (2.1) – (2.4), we obtain a system of six linear homogeneous equations relative to the constants C_i, D_i (for the continuous cylinder there will be three equations); these systems will have nonzero solutions if the corresponding determinants are equal to zero. The transcendental equation obtained as a result of expanding such a determinant, defines the proper values of the corresponding homogeneous problem, i.e. the parameters λ_i.

Thus, for the continuous cylinder $(a = 1,\ b = 0)$ we have the following transcendental equations, corresponding to the cases (1) – (4):

$$\begin{aligned}&[\lambda^6 + 3n^2\lambda^4 + (3n^2 - 3 + 2\nu)\, n^2\lambda^2 + n^4(n^2 - 1)]\, I_n^3(\lambda) - 2\,[\lambda^4 - \\ &(1 - 2\nu)\, n^2\lambda^2 - 2(1 - \nu)\, n^2(n^2 - 1)]\, \lambda I_n^2(\lambda)\, I_n'(\lambda) - [\lambda^4 + 2\,(n^2 + 1 - \\ &\nu)\, \lambda^2 + n^2(n^2 - 1)]\, \lambda^2 I_n(\lambda)\, I_n'^2(\lambda) + 2\,[\lambda^2 - 2\,(1 - \nu)\,(n^2 - \\ &1)]\, \lambda^3 I_n'^3(\lambda) = 0\end{aligned} \tag{2.5}$$

$$\begin{aligned}&4\,(1 - \nu)\, n^2 I_n^3(\lambda) + (\lambda^2 + n^2)\, \lambda I_n^2(\lambda)\, I_n'(\lambda) - \\ &4\,(1 - \nu)\, \lambda^2 I_n(\lambda)\, I_n'^2(\lambda) - \lambda^3 I_n'^3(\lambda) = 0\end{aligned} \tag{2.6}$$

$$\begin{aligned}&(\lambda^2 + n^2)\, n^2 I_n^3(\lambda) + 4\,(1 - \nu)\, \lambda I_n^2(\lambda)\, I_n'(\lambda) + \\ &[2\,(1 - \nu)\, \lambda^2 - n^2]\, \lambda^2 I_n(\lambda)\, I_n'^2(\lambda) - 4\,(1 - \nu)\, \lambda^3 I_n'^3(\lambda) = 0\end{aligned} \tag{2.7}$$

$$\begin{aligned}&4\,(1 - \nu)\, n^2 I_n^3(\lambda) + [(3 - 2\nu)\, \lambda^2 + n^2]\, \lambda I_n^2(\lambda)\, I_n'(\lambda) - \\ &4\,(1 - \nu)\, \lambda^2 I_n(\lambda)\, I_n'^2(\lambda) - \lambda^3 I_n'^3(\lambda) = 0\end{aligned} \tag{2.8}$$

Equation (2.5) is given also in [4].

We prove the following orthogonality relation:

$$J \equiv \int_b^a (u_j^*\tau_k^* + v_j^*\theta_k^* - \sigma_j^* w_k^*)\, r\,dr = 0 \qquad (j \neq k) \tag{2.9}$$

Here the quantities u^*, v^*, w^*, $\tau^* = \tau_{zr}^*$, $\theta^* = \tau_{z\varphi}^*$, $\sigma^* = \sigma_z^*$ are determined by the formulas (1.6), (1.7); the subscripts j and k correspond to the proper values λ_j^2 and λ_k^2. For the proof we insert the expressions (1.6), (1.7) into the integral (2.9) and we transform the third term by integration by parts

$$-\int_b^a \sigma_j^* w_k^* r dr = 2G\lambda_k \int_b^a \left[\frac{n^2}{r}\omega_j + n\chi_j + (r\psi_j - r\omega_j')'\right]\omega_k dr =$$

$$2G\lambda_k \left\{[r(\psi_j - \omega_j')\,\omega_k]_b^a + \int_b^a \left[\left(\frac{n^2}{r}\omega_j + n\chi_j\right)\omega_k + (\omega_j' - \psi_j)\,\omega_k' r\right] dr\right\} \quad (2.10)$$

In the formula (2.10) the term which is not under the integral signs vanishes for any of the conditions (2.1) – (2.4), and consequently, the integral (2.9) takes the form

$$J = 2G\lambda_k I \quad (2.11)$$

$$I = \int_b^a \left[\psi_j\omega_k' + \omega_j'\psi_k - \frac{n}{r}(\chi_j\omega_k + \omega_j\chi_k) - 2(\psi_j\psi_k + \chi_j\chi_k)\right] r dr$$

Let us consider the expression $\lambda_k^2 I$. We integrate the first term by parts

$$\lambda_k^2 \int_b^a \psi_j\omega_k' r dr = \lambda_k^2 [\psi_j\omega_k r]_b^a - \lambda_k^2 \int_b^a (\psi_j r)'\omega_k dr \quad (2.12)$$

In the remaining terms and in the integral in the right-hand side of the relation (2.12), we eliminate the products $\lambda_k^2\psi_k$, $\lambda_k^2\chi_k$, $\lambda_k^2\omega_k$ with the aid of the differential equations (1.5). After a series of computations we obtain

$$\begin{aligned}\lambda_k^2 I = \int_b^a \Big\{ & \frac{1}{(1-\nu)\,r}[(\psi_j r)'(\psi_k r)' + n^2\chi_j\chi_k + n(\psi_j r)'\chi_k + n\chi_j(\psi_k r)'] + \\ & \frac{n^3}{r^2}(\omega_j\chi_k + \chi_j\omega_k) + 2r(\psi_j'\psi_k' + \chi_j'\chi_k') + \omega_j'\left[\frac{(\psi_k r)'}{r}\right]' + \\ & \left[\frac{(\psi_j r)'}{r}\right]'\omega_k' + nr\left[\omega_j'\left(\frac{\chi_k}{r}\right)' + \left(\frac{\chi_j}{r}\right)'\omega_k'\right] + \frac{4n}{r}(\psi_j\chi_k + \chi_j\psi_k) - \\ & n^2\left[\psi_j\left(\frac{\omega_k}{r}\right)' + \left(\frac{\omega_j}{r}\right)'\psi_k\right] + \frac{2}{r}(n^2+1)(\psi_j\psi_k + \chi_j\chi_k)\Big\} dr + \\ & \left[\psi_j\left(\lambda_k^2 r + \frac{n^2}{r}\right)\omega_k - \{n\chi_j + (\psi_j r)'\}\,\omega_k' - \frac{n}{r}\omega_j(\chi_k r)' - 2r(\psi_j\psi_k' + \chi_j\chi_k')\right]_b^a\end{aligned} \quad (2.13)$$

In order to obtain the result (2.13) it is necessary to perform the integration by parts of the following expression:

$$\int_b^a \Big[\frac{n^2}{r}\psi_j'\omega_k - n\left(\chi_j\omega_k'' + \omega_j\chi_k'' + \frac{\omega_j'\chi_k}{r}\right) -$$
$$(\psi_j r)'\omega_k'' - 2\psi_j(\psi_k' r)' - 2\chi_j(\chi_k' r)'\Big] dr$$

We note that the value of the integrals (2.13) does not change at the transposition of the subscripts j and k. We interchange now in (2.13) the subscripts j and k and we subtract the obtained expression from (2.13) ; the integrals drop out and the sum of the terms outside the integral signs vanish for the homogeneous boundary conditions (2.1)–(2.4). Hence it follows that $I = 0$ for $j \neq k$. By virtue of (2.11), the desired orthogonality relation (2.9) is proved.

The relation (2. 9) allows us to obtain the exact solution of the problem of the equilibrium of the cylinder in the case when at the ends $z = \pm l$ of the cylinder there are given either the normal stresses and the tangential displacements (σ_z, u, v) or the shear stresses and the normal displacements $(\tau_{zr}, \tau_{z\varphi}, w)$.

3. As an example we consider the first version of the end conditions and for the sake of simplicity we will assume that they are symmetric with respect to the plane $z = 0$ (the antisymmetric case is examined in a similar manner). Expanding the end values of the given quantities in series with respect to the angle φ, we obtain

$$u(r, \varphi, \pm l) = \sum_{n=0}^{\infty} u_n{}^l(r) \cos n\varphi, \qquad v(r, \varphi, \pm l) = \sum_{n=1}^{\infty} v_n{}^l(r) \sin n\varphi$$

$$\sigma_z(r, \varphi, \pm l) = \sum_{n=0}^{\infty} \sigma_{zn}{}^l(r) \cos n\varphi \tag{3.1}$$

Forming the series from the solutions (1. 3), whose parameters λ_j are determined by one of the conditions (2. 1) – (2. 4), we have

$$u(r, \varphi, z) = \sum_{j=1}^{\infty} \sum_{n=0}^{\infty} A_{jn} u_{jn}{}^*(r) \cos n\varphi \cos \lambda_j z$$

$$v(r, \varphi, z) = \sum_{j=1}^{\infty} \sum_{n=1}^{\infty} A_{jn} v_{jn}{}^*(r) \sin n\varphi \cos \lambda_j z$$

$$\sigma_z(r, \varphi, z) = \sum_{j=1}^{\infty} \sum_{n=0}^{\infty} A_{jn} \sigma_{jn}{}^*(r) \cos n\varphi \cos \lambda_j z \tag{3.2}$$

where A_{jn} are arbitrary constants, subject to determination from the boundary conditions at the ends ; in the given problem the latter have the following form :

$$\sum_{j=1}^{\infty} A_{jn} u_{jn}{}^*(r) \cos \lambda_j l = u_n{}^l(r)$$

$$\sum_{j=1}^{\infty} A_{jn} v_{jn}{}^*(r) \cos \lambda_j l = v_n{}^l(r)$$

$$\sum_{j=1}^{\infty} A_{jn} \sigma_{jn}{}^*(r) \cos \lambda_j l = \sigma_{zn}{}^l(r) \tag{3.3}$$

The constants A_{jn} can be found from Eqs. (3. 3), if we make use of the relation (2. 9). We multiply the first equation of (3. 3) by $r\tau_{kn}{}^*(r)$, the second one by $r\theta_{kn}{}^*(r)$, the third one by $rw_{kn}{}^*(r)$, we add the obtained products and we integrate with respect to r from b to a. Then we are left only with the terms for which the subscript $j = k$ by virtue of the generalized orthogonality relation, and the desired constants have the form

$$A_{kn} = \frac{1}{J_k \cos \lambda_k l} \int_b^a (u_n{}^l \tau_{kn}{}^* + v_n{}^l \theta_{kn}{}^* - \sigma_{zn}{}^l w_{kn}{}^*)\, r dr \tag{3.4}$$

where J_k is the value of the integral (2. 9) for $j = k$.

In order to complete the analysis we obtain the expression of the quantity J_k. From the formulas (2. 11), (2. 13) we have

$$J_k = 2G\lambda_k \lim_{\lambda\to\lambda_k} \frac{1}{\lambda^2-\lambda_k^2}\Big[\Big(\lambda^2 r + \frac{n^2}{r}\Big)\omega\psi_k + \frac{n}{r}\,\omega\,(\chi_k r)' +$$
$$n\chi\omega_k' + (\psi r)'\,\omega_k' + 2r\psi\psi_k' + 2r\chi\chi_k' - \psi\Big(\lambda_k^2 r + \frac{n^2}{r}\Big)\omega_k -$$
$$\frac{n}{r}\,(\chi r)'\,\omega_k - \omega'\{n\chi_k + (\psi_k r)'\} - 2r\psi'\psi_k - 2r\chi'\chi_k\Big]_b^a \qquad (3.5)$$

Removing by l'Hospital's rule, the indeterminacy in the expression (3.5) and making use of the formula for differentiation with respect to the parameter λ of the functions ψ, χ, ω and of their derivatives, we obtain, after taking the limit,

$$J_k = \frac{2G}{\lambda_k}\Big[(\lambda_k^2 r^2 + n^2)\Big\{\psi_k\omega_k' - (\psi_k r)'\frac{\omega_k}{r} - n\frac{\omega_k}{r}\chi_k - \psi_k^2 - \chi_k^2\Big\} -$$
$$\frac{1}{2(1-\nu)}\{(\psi_k r)' + n\chi_k\}^2 + (\psi_k r)'\,\omega_k' + (\lambda_k^2 r^2 - n^2)\,\psi_k\frac{\omega_k}{r} +$$
$$(\psi_k' r)^2 - \psi_k^2 + (\chi_k' r)^2 - \chi_k^2 + n\Big\{\chi_k\omega_k' + (\chi_k r)'\Big(\omega_k' - \frac{\omega_k}{r}\Big) - 4\psi_k\chi_k\Big\}\Big]_b^a \qquad (3.6)$$

For the computation of the expression (3.6) we have made use of Eqs (1.5). We obtain concrete expressions for the quantities J_k in the case of the homogeneous boundary conditions (2.1) – (2.4) at the surfaces of the cylinder if we take into account the equalities (2.1) – (2.4); we obtain, respectively,

$$J_k = \frac{2G}{\lambda_k}\Big[\lambda_k^2 r^2\Big(\frac{\psi_k\omega_k}{r} - \chi_k^2\Big) - \frac{1}{2(1-\nu)}\Big\{\psi_k + 2n\chi_k + (\lambda_k^2 r^2 + n^2)\frac{\omega_k}{r}\Big\}^2\Big]_b^a$$

$$J_k = \frac{2G}{\lambda_k}\Big[\lambda_k^2 r^2\psi_k^2 + (n\psi_k + \chi_k' r)^2 + \frac{1-2\nu}{2(1-\nu)}\{(\psi_k r)'\}^2\Big]_b^a$$

$$J_k = \frac{2G}{\lambda_k}\Big[(\psi_k' r)^2 - (\lambda_k^2 r^2 + n^2)\Big(\omega_k\psi_k' + \frac{n}{r}\chi_k\omega_k + \chi_k^2\Big) - \frac{1}{2(1-\nu)}(r\psi_k' + n\chi_k)^2\Big]_b^a$$

$$J_k = \frac{2G}{\lambda_k}\Big[(\chi_k' r)^2 + nr\omega_k'\chi_k' + (\lambda_k^2 r^2 + n^2)\,\psi_k(\omega_k' - \psi_k) - \frac{1}{2(1-\nu)}(2\psi_k - \omega_k')^2\Big]_b^a$$

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